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# STABILITY OF THE CONTROL OF A PLANE TURN OF A SPACECRAFT*

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A method of extensive control based on a closed scheme is discussed, and the possibility of constructing a stabilizing momentum ensuring the asymptotic stability of the programmed motion of a spacecraft is proved.

The method of reorienting a non-symmetric spacecraft at rest by means of a single turn in a plane about a fixed axis, is called extensive /1/ and yields a payoff in high-speed action and in energy consumption. The problems of constructing optimal control programs based on the above method for various specific functionals were solved in /2-4/. The instability of the resulting program solutions however reduces their practical value.

1. Let us introduce two coordinate systems with a common origin at the centre of mass of the craft, the inertial system $X_1X_2X_3$ and the system $x_1x_2x_3$ rigidly connected with the craft. We assume that at the initial instant $(t = 0)$ the corresponding axes of the systems coincide, and the required reorientation of the craft is executed by its rotation about an axis whose direction is given by the unit vector

$$\mathbf{v} = \sum_{k=1}^{3} \nu_k e_k = \sum_{k=1}^{3} \nu_k i_k$$

where $\nu_k$ are the direction cosines, $e_k$ and $i_k$ are unit vectors of the axes $X_k$ and $x_k$, with the angle of rotation $\varphi_0 \leqslant \pi$ given. The motion of the craft about the centre of mass is described by Euler's equations, which have the following form when the axes $x_k$ $(k = 1, 2, 3)$ coincide with the principal axes of inertia:

$$I_1\omega_1' + (I_3 - I_2)\,\omega_2\omega_3 = \mathrm{M}_1 \qquad (1\ 2\ 3) \tag{1.1}$$

Here $I_k$ are the principal moments of inertia of the craft, $\omega_k$ and $\mathrm{M}_k$ are projections of the vectors $\boldsymbol{\omega}$ and $\mathbf{M}$ of angular velocity and the external force moment respectively onto the axes $x_k$ $(k = 1, 2, 3)$, and a prime denotes a derivative with respect to time. Using the Rodrigues-Hamilton parameters /5/ to determine the orientation, we obtain the kinematic equations in the form

$$\begin{gathered} 2\lambda_0' = -\omega_1\lambda_1 - \omega_2\lambda_2 - \omega_3\lambda_3 \\ 2\lambda_1' = \omega_1\lambda_0 + \omega_3\lambda_2 - \omega_2\lambda_3 \qquad (1\ 2\ 3) \end{gathered} \tag{1.2}$$

Eqs.(1.1) and (1.2) together form a complete system of equations for solving the problem of controlling the turn of the spacecraft.

Programmed control using an open scheme is carried out under the assumption that the angular velocity vector of the craft coincides with the given direction of the $\mathbf{v}$ axis:

$$\boldsymbol{\omega} = \omega\mathbf{v} = \varphi'\mathbf{v} \tag{1.3}$$

where $\omega(t)$ is the angular velocity and $\varphi(t)$ is the angle of rotation at the instant $t$. Taking into account (1.3), we obtain the following expression for the vector of the programmed control force moment:

$$J_\alpha\varphi''\mathbf{e}_\alpha + J_\beta\varphi'^2\mathbf{e}_\beta = \mathbf{M} \tag{1.4}$$

The unit vectors $\mathbf{e}_\alpha$ and $\mathbf{e}_\beta$ introduced here as well as the parameter $J_\alpha$ and $J_\beta$, are given in terms of the inertial tensor I as follows:

$$J_\alpha = |\mathbf{Iv}|, \quad J_\beta = |\mathbf{v} \times \mathbf{Iv}|, \quad \mathbf{e}_\alpha = \mathbf{Iv}/J_\alpha, \quad \mathbf{e}_\beta = \mathbf{v} \times \mathbf{Iv}/J_\beta$$

Thus, if the function $\varphi(t)$ of programmed change in the angle of rotation is known, the vector of control of the force moment can be found from (1.4).

2. The programmed rotational motion of the craft is unstable with respect to the initial perturbations, therefore the plane rotation can be controlled using a closed scheme. When external information concerning the running angular position and angular velocity of the craft is available, the control can be constructed by comparing the programmed motion with the

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actual motion. In this case the basic problem will consist of choosing a technically realizable structure of the stabilizing moment, beginning with the requirement to ensure the asymptotic stability of the programmed motion. To solve this problem we denote the parameters of the actual motion of the craft by $\omega^*(t)$, $\Lambda^*(t)$, and in this case we shall have

$$\omega_k^* = \omega_k(t) + \Omega_k, \quad \omega_k(t) = \omega(t)\nu_k$$
$$\Lambda^* = (\lambda_0^*, \lambda_1^*, \lambda_2^*, \lambda_3^*), \quad \lambda_0^* = \lambda_0 + \xi_0$$
$$\lambda_k^* = \lambda_k + \xi_k, \ M_k^* = M_k + m_k, \quad k = 1, 2, 3 \tag{2.1}$$

where $\Omega_k$, $\xi_0$, $\xi_k$ are the perturbations, $\omega_k$, $\lambda_0$, $\lambda_k$, $M_k$ are the programmed values of the components of the angular velocity, quaternion, and control moment, and $m_k$ $(k = 1, 2, 3)$ are the components of the required stabilizing force moment. Substituting expressions (2.1) into Eq.(1.1) and (1.2), we obtain the following equations of perturbed motion:

$$I_1\Omega_1' + (I_3 - I_2)(\omega\nu_3\Omega_2 + \omega\nu_2\Omega_3 + \Omega_2\Omega_3) = m_1 \quad (1\ 2\ 3) \tag{2.2}$$
$$2\xi_0' = -\nu_1\Omega_1 \sin\frac{\varphi}{2} - \nu_2\Omega_2 \sin\frac{\varphi}{2} - \nu_3\Omega_3\sin\frac{\varphi}{2} - \omega(\nu_1\xi_1 + \nu_2\xi_2 + \nu_3\xi_3) -$$
$$\Omega_1\xi_1 - \Omega_2\xi_2 - \Omega_3\xi_3$$
$$2\xi_1' = \Omega_1\cos\frac{\varphi}{2} + \nu_2\Omega_3\sin\frac{\varphi}{2} - \nu_3\Omega_2\sin\frac{\varphi}{2} + \omega(\nu_1\xi_0 + \nu_3\xi_2 - \nu_2\xi_3) +$$
$$\Omega_1\xi_0 + \Omega_3\xi_2 - \Omega_2\xi_3 \quad (1\ 2\ 3)$$

We shall assume that the components of the stabilizing force moment are linear functions of the perturbations of kinematic parameters

$$m_k = -\gamma\theta_k - \delta_k\Omega_k \quad (k = 1, 2, 3) \tag{2.3}$$

where $\gamma$, $\delta_k$ are positive constants and $\theta_k$ is the angular deviation of the programmed motion from the actual motion. We will express the perturbation of the angular position $\theta = \theta_1 i_1 + \theta_2 i_2 + \theta_3 i_3$ using the quaternions of the programmed motion $\Lambda$ and perturbed motion $\Lambda^* - \Lambda$ of the vector part of their quaternion product

$$\theta = \operatorname{vect}(\Lambda_1 - \Lambda_2) \circ \Lambda \tag{2.4}$$

where $\Lambda_1$ and $\Lambda_2$ are the quaternions conjugated, respectively, with $\Lambda$ and $\Lambda^*$. Taking the relations (2.1) into account we obtain $\theta$, and substituting the values of $\theta_k$ into (2.3) we obtain the following expressions for the stabilizing moments:

$$m_1 = -\gamma\left(-\nu_1\xi_0\sin\frac{\varphi}{2} - \xi_1\cos\frac{\varphi}{2} + \nu_3\xi_2\sin\frac{\varphi}{2} - \nu_2\xi_3\sin\frac{\varphi}{2}\right) - \delta_1\Omega_1 \quad (1\ 2\ 3) \tag{2.5}$$

Let us consider the positive-definite function

$$V = (I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2)/2 + \gamma(\xi_0^2 + \xi_1^2 + \xi_2^2 + \xi_3^2) \tag{2.6}$$

The time derivative of the function (2.6), by virtue of the system (2.2) taking (2.5) into account, has the form

$$V' = -\delta_1\Omega_1^2 - \delta_2\Omega_2^2 - \delta_3\Omega_3^2 - (I_1 - I_2)\nu_3\omega\Omega_1\Omega_2 +$$
$$(I_1 - I_3)\nu_2\omega\Omega_1\Omega_3 - (I_2 - I_3)\nu_1\omega\Omega_2\Omega_3 \tag{2.7}$$

Assuming, without any loss of generality, that $I_1 \geqslant I_2 \geqslant I_3$, we can reduce the condition of constant negativity of the function (2.7) to a single inequality

$$\delta_1\delta_2\delta_3 - {}^1/_4(I_1 - I_2)(I_2 - I_3)(I_1 - I_3)\nu_1\nu_2\nu_3\omega_m^2 - {}^1/_4(I_1 - I_3)^2\nu_2^2\omega_m^2\delta_2 -$$
$${}^1/_4(I_1 - I_2)^2\nu_3^2\omega_m^2\delta_3 - {}^1/_4(I_2 - I_3)^2\nu_1^2\omega_m^2\delta_1 > 0$$
$$\omega_m = \max_{t\geqslant 0}|\omega(t)| \tag{2.8}$$

When the inequality (2.8) holds, the unperturbed motion

$$\omega_k = \omega(t)\nu_k, \ \lambda_0 = \cos\varphi/2, \ \lambda_k = \nu_k\sin\varphi/2$$

will be stable with respect to $\Omega_k$, $\xi_0$, $\xi_k$ $(k = 1, 2, 3)$. We shall show that it is asymptotically stable. With this purpose in mind, we shall consider another function

$$W = I_1\Omega_1\chi_1 + I_2\Omega_2\chi_2 + I_3\Omega_3\chi_3 \tag{2.9}$$
$$\chi_1 = -\xi_1\cos\frac{\varphi}{2} + (\nu_1\xi_0 + \nu_2\xi_3 - \nu_3\xi_2)\sin\frac{\varphi}{2} \quad (1\ 2\ 3)$$

The time derivative of the function (2.9) is equal on the set $\{V' = 0: \Omega_k = 0, \ k = 1, 2, 3\}$, by virtue of system (2.2), to

$$W' = \gamma\,(\chi_1{}^2 + \chi_2{}^2 + \chi_3{}^2) \tag{2.10}$$

We note that the perturbations $\xi_0, \xi_k \ (k = 1, 2, 3)$ satisfy the equation

$$\xi_0{}^2 + \xi_1{}^2 + \xi_2{}^2\xi_2{}^2 + \xi_3{}^2 + 2\xi_0 \cos \varphi/2 + 2(\nu_1\xi_1 + \nu_2\xi_2 + \nu_3\xi_3) \sin \varphi/2 = 0 \tag{2.11}$$

We shall show that $W'$ is a positive definite function of the variable $\xi_0, \xi_k$. The equation $W' = 0$ is equivalent to the system

$$\chi_1 = \chi_2 = \chi_3 = 0 \tag{2.12}$$

If the function $\omega\,(t)$ is such that

$$\omega\,(0) = 0, \quad \varphi\,(0) = 0, \quad \varphi = \int_0^t \omega\,(\tau)\,d\tau$$

and $\varphi\,(t)$ does not approach π infinitely closely on some interval $0 < t < t_1$, then expressing $\xi_k \ (k = 1, 2, 3)$ in terms of $\xi_0$ from the system (2.1) according to the formulas

$$\xi_k = \nu_k \xi_0 \,\mathrm{tg}\,(\varphi/2) \tag{2.13}$$

and substituting expressions (2.13) into (2.11), we obtain

$$\xi_0 \sec^2(\varphi/2)\,(\xi_0 + 2\cos(\varphi/2)) = 0$$

The above expression implies $\xi_0 = 0$, and from (2.13) we have $\xi_k = 0 \ (k = 1, 2, 3)$. The function $\varphi\,(t)$, in some interval $t_1 < t < t_2$, takes the values from a sufficiently small neighbourhood of π. In this case we can use system (2.12) to express $\xi_0, \xi_1, \xi_2$ in terms of $\xi_3$:

$$\xi_0 = \xi_3 \,\mathrm{ctg}\,(\varphi/2)/\nu_3, \ \xi_1 = \nu_1\xi_3/\nu_3, \ \xi_2 = \nu_2\xi_3/\nu_3 \tag{2.14}$$

Substituting expressions (2.14) into (2.11), we obtain

$$\xi_3 \,\mathrm{cosec}\,(\varphi/2)\,(\xi_2 \,\mathrm{cosec}\,(\varphi/2) + 2) = 0$$

The above relation implies that $\xi_3 = 0$, and (2.14) implies that $\xi_0 = \xi_1 = \xi_2 = 0$. By virtue of Theorem 1.2 of /6/ the unperturbed motion is asymptotically stable with respect to the variables $\Omega_k, \xi_0, \xi_k \ (k = 1, 2, 3)$ when the inequality (2.8) holds.

The results obtained establish the possibility in principle of constructing an asymptotically stable closed system of controlling plane rotations when the programmed motion is assumed known.

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